Chapter 12: Sequences & Series

Definitions of a Sequence

An infinite sequence \( \{a_n\} \) is a function whose domain is the set of positive integers. The function values, or terms, of the sequence are represented by

\[
a_1, a_2, a_3, a_4, \ldots a_n, \ldots
\]

Sequences whose domains consist of only the first \( n \) positive integers are called finite sequences.

A Famous Infinite Sequence: Fibonacci Sequence

\[
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots
\]

Symbol

\[
a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8
\]

Term

1, 1, 2, 3, 5, 8, 13, 21

Write the first four terms of the sequence whose \( n \)th term, of general term, is given:

a. \( a_n = 3n + 4 \)

\[
\begin{align*}
a_1 &= 3(1)+4 = 7 \\
a_2 &= 3(2)+4 = 10 \\
a_3 &= 3(3)+4 = 13 \\
a_4 &= 3(4)+4 = 16
\end{align*}
\]

b. \( a_n = (-1)^n \frac{3^n - 1}{3^3 - 1} \)

\[
\begin{align*}
a_1 &= \frac{(-1)^1}{3^1 - 1} = \frac{-1}{2} \\
a_2 &= \frac{(-1)^2}{3^2 - 1} = \frac{1}{8} \\
a_3 &= \frac{(-1)^3}{3^3 - 1} = \frac{-1}{26} \\
a_4 &= \frac{(-1)^4}{3^4 - 1} = \frac{1}{80}
\end{align*}
\]
Graph of a Sequence
Set of discrete points
Thus not connected by a line

1. Graph the function
   \( f(x) = 2x + 3 \)

2.) Graph the sequence whose general term in \( a_n = 2n + 3 \)

Using a recursive function

Recursive Function - defines the \( n \)th term of a sequence as a function of the previous term.

1.) Find the first four terms of the sequence in which
   \( a_1 = 5 \) and \( a_n = 3a_{n-1} + 2 \) for \( n \geq 2 \)
   
   - \( a_1 = 5 \)
   - \( a_2 = 3 \cdot 5 + 2 = 17 \)
   - \( a_3 = 3 \cdot 17 + 2 = 53 \)
   - \( a_4 = 3 \cdot 53 + 2 = 161 \)

2.) Find the first four terms of the sequence in which
   \( a_1 = 3 \) and \( a_n = 2a_{n-1} + 5 \) for \( n \geq 2 \)
   
   - \( a_1 = 3 \)
   - \( a_2 = 2 \cdot 3 + 5 = 11 \)
   - \( a_3 = 2 \cdot 11 + 5 = 27 \)
   - \( a_4 = 2 \cdot 27 + 5 = 59 \)
Factorial Notation

If \( n \) is a positive integer, the notation \( n! \) (read "\( n \) factorial") is the product of all the positive integers from \( n \) down through 1.

\[ n! = n(n-1)(n-2)\ldots(3)(2)(1) \]

\( 0! \) (zero factorial), by definition, is 1.

Find each value.
1.) \( 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \)
2.) \( 2.3! = 2 \cdot 3 \cdot 2 \cdot 1 = 12 \)
3.) \( (2.3)! = 6! = 720 \)
4.) \( 3!2! = 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 = 12 \)

Find the first four terms of the sequence whose \( n \)th term is

\[ a_n = \frac{20}{(n+1)!} \]

\[ a_1 = \frac{20}{2!} = \frac{20}{2 \cdot 1} = 10 \]
\[ a_2 = \frac{20}{3!} = \frac{20}{6} = \frac{10}{3} \]
\[ a_3 = \frac{20}{(4+1)!} = \frac{20}{4!} = \frac{5}{6} \]
\[ a_4 = \frac{20}{(5+1)!} = \frac{20}{5!} = \frac{20}{120} = \frac{1}{6} \]

Evaluating Fractions with Factorials
1.) \( \frac{26!}{21!} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21!}{21!} = 78 \)
2.) \( \frac{10!}{2!8!} = \frac{10 \cdot 9 \cdot 8!}{2 \cdot 1 \cdot 8!} = 45 \)
3.) \( \frac{14!}{2!12!} = \frac{14 \cdot 13 \cdot 12!}{2 \cdot 1 \cdot 12!} = 91 \)
4.) \( \frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n+1 \)
**Summation Notation**

It is sometimes useful to find the sum of the first $n$ terms of a sequence, called a series.

The notation used to express a sum is:

$$
\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n
$$

### Example:

1. \(\sum_{i=1}^{6} (i^2 + 1)\)

   - \(a_1 = (1^2 + 1) = 2\)
   - \(a_2 = (2^2 + 1) = 5\)
   - \(a_3 = (3^2 + 1) = 10\)
   - \(a_4 = (4^2 + 1) = 17\)
   - \(a_5 = (5^2 + 1) = 26\)
   - \(a_6 = (6^2 + 1) = 37\)

   \[\sum_{i=1}^{6} (i^2 + 1) = 2 + 5 + 10 + 17 + 26 + 37 = 97\]

2. \(\sum_{k=4}^{7} [(-2)^k - 5]\)

   - \(a_4 = (-2)^4 - 5 = 1\)
   - \(a_5 = (-2)^5 - 5 = -37\)
   - \(a_6 = (-2)^6 - 5 = -133\)
   - \(a_7 = (-2)^7 - 5 = -133\)

   \[\sum_{k=4}^{7} [(-2)^k - 5] = 1 - 37 + 59 - 133 - 133 = -100\]
Writing Series in Summation Notation

3. \[ \sum_{i=1}^{5} 3 \]

\[ a_1 = 3 \]
\[ a_2 = 3 \]
\[ a_3 = 3 \]
\[ a_4 = 3 \]
\[ a_5 = 3 \]
\[ \sum_{i=1}^{5} 3 = 15 \]

4. \[ \sum_{i=2}^{4} 2i^2 \]

\[ a_2 = 2 \cdot 2^2 = 8 \]
\[ a_3 = 2 \cdot 3^2 = 18 \]
\[ a_4 = 2 \cdot 4^2 = 32 \]
\[ \sum_{i=2}^{4} 2i^2 = 58 \]

1. \[ 1^3 + 2^3 + 3^3 + \ldots + 7^3 \]

\[ \sum_{i=1}^{7} i^3 \]

2. \[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^{n-1}} \]

\[ \sum_{i=1}^{n} \frac{1}{3^{i-1}} \]